ON THE NUMBER OF ZEROS OF CERTAIN RATIONAL HARMONIC FUNCTIONS

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Abstract. Extending a result from [KS 03], we show that the rational harmonic function $\overline{r(z)} - z$, where $r(z)$ is a rational function of degree $n > 1$, has no more than $5n - 5$ complex zeros. Applications to gravitational lensing are discussed. In particular, this result settles a conjecture [Rh 01] concerning the maximum number of lensed images due to an n-point gravitational lens.

1. Introduction

A. Wilmshurst [Wil 98] showed that there is an upper bound on the number of zeros of a harmonic polynomial $f(z) = p(z) - \overline{q(z)}$, where p and q are analytic polynomials of different degree, answering the question of T. Sheil-Small [SS 92]. Let $n = \deg p > \deg q = m$. Wilmshurst showed that n^2 is a sharp upper bound when $m = n - 1$ and conjectured that the upper bound is actually $m(m-1) + 3n - 2$. D. Khavinson and G. Świątek [KS 03] showed that Wilmshurst's conjecture holds for the case $n > 1, m = 1$ using methods from complex dynamics. When hearing of this result, P. Poggi-Corradini asked whether this approach can be extended to the case $f(z) = p(z)/q(z) - \overline{z}$, where p and q are analytic polynomials.

In this note, we apply the approach from [KS 03] to prove

Theorem 1. *Let $r(z) = p(z)/q(z)$ be a rational function where p and q are relatively prime, analytic polynomials and such that $n = \deg\ r = \max(\deg\ p, \deg\ q) > 1$. Then*

$$\#\{z \in \mathbb{C} : \overline{r(z)} = z\} \leq 5n - 5$$

We note that the zeros of $\overline{r(z)} - z$ are isolated, because each zero is also a fixed point of $Q(z) = \overline{r(\overline{r(z)})}$, an analytic rational function of degree n^2. This also follows from a result of P. Davis [Da 74] (Chapter 14) concerning the Schwarz functions of analytic curves. (The Schwarz function is an analytic function $S(z)$ that gives the equation of a curve in the form $\overline{z} = S(z)$, *cf.* [Da 74].) A rational Schwarz function implies that the curve is a line or a circle, so the degree must be one.

We also note that $\overline{r(z)} - z$ will not have a zero at ∞. If ∞ were a zero, then $r(z) = az + b + O(1/z)$. By the change of variable $z = 1/w$, we see that $a/w - 1/\overline{w} + b + O(w)$ has a zero at $w = 0$. Restricting w to the real axis, we see that $a = 1$ and $b = 0$. We obtain a contradiction when we restrict w to the imaginary axis.

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We now discuss an application of our result to gravitational microlensing. An n-point gravitational lens can be modeled as follows. Suppose that we have n point masses (such as stars). Construct a plane through the center of mass of these point masses, such that the line of sight from the observer to the center of mass is orthogonal to this plane. This plane is called the lens plane (or deflector plane). Suppose that the lens plane is between the observer and a light source. (We are assuming that the distance between the point masses is small compared to the distance between the observer and the lens plane, as well as the distance between the lens plane and the light source.) The plane containing our light source which is parallel to the lens plane is called the source plane. Due to the deflection of light by the point masses, multiple images of the light source are formed. This phenomenon is known as gravitational microlensing. See [Wa 98] and [NB 96] for an introduction to gravitational lensing and [St 97] for an introduction to a complex formulation of lensing theory; also see [PLW 01].

Gravitational microlensing can be modeled by a lens equation, which defines a mapping from the lens plane to the source plane. To set up a lens equation for our n-point gravitational lens, the point masses are projected onto the lens plane. The projection of the j-th point mass has a scaled mass of m_j and is located at a scaled coordinate of z_j in the lens plane, where m_j is a positive constant and z_j is a complex constant. Suppose that we have a light source located at a scaled coordinate of w in the source plane. Following [Wit 90], this lens equation will be given by $w = z + \gamma\overline{z} - \text{sign}(\sigma)\Sigma_{j=1}^{n} m_j/(\overline{z} - \overline{z_j})$, where the normalized (external) shear γ and the optical depth (or normalized surface density) $\sigma \neq 0$ are real constants. In this model, if z satisfies the lens equation, then our gravitational lens will map z to w; hence z corresponds to the position of a lensed image. The number of lensed images is the number of solutions of the lens equation.

We can rewrite this lens equation in terms of the rational harmonic function $f(z) = \overline{r(z)} - z$ by letting $r(z) = \overline{w} - \gamma z + \text{sign}(\sigma)\Sigma_{j=1}^{n} m_j/(z - z_j)$. We thus see that the zeros of $f(z)$ are solutions of the lens equation for a light source at position w. H. Witt [Wit 90] showed for $n > 1$ that the maximum number of observed images is at most $n^2 + 1$ when $\gamma = 0$ and $(n+1)^2$ when $\gamma \neq 0$. A. Petters [Pe 92] used Morse theory to obtain further estimates for both cases. S. Mao, A. Petters, and H. Witt [MPW 97] conjectured that the maximum number of images is linear in n for the case $\gamma = 0$ and $\sigma > 0$. S. H. Rhie [Rh 01] later conjectured that for $n > 1$ such a gravitational lens gives at most $5n - 5$ images. In the $\gamma = 0$ case, deg $r = n$ and Theorem 1 settles this conjecture. Further, for the case $\gamma \neq 0$, we see that deg $r = n + 1$, so Theorem 1 gives an upper bound of $5(n + 1) - 5 = 5n$ lensed images. As a result, we have the following corollary:

Corollary 1. *An n-point gravitational lens modeled by $w = z + \gamma\overline{z} - sign(\sigma)\Sigma_{j=1}^{n} m_j/(\overline{z} - \overline{z_j})$, where $n > 1$ can produce at most $5n - 5$ images when the shear $\gamma = 0$ and at most $5n$ images when the shear $\gamma \neq 0$.*

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3. Preliminaries

We first recall some terminology and a fact from complex dynamics. Let $r(z) = p(z)/q(z)$ be a rational function, where p and q are relatively prime, analytic polynomials. The degree of r, denoted by $\deg r$, is given by $\max(\deg p, \deg q)$. z is called a *critical point of* r if $r'(z) = 0$. We will be interested in counting the fixed points of a rational function. A fixed point $z_0 \in \mathbb{C}_\infty$ is said to be *attractive*, *repelling* or *neutral* if $|r'(z_0)| < 1$, $|r'(z_0)| > 1$, or $|r'(z_0)| = 1$ respectively. A neutral fixed point where the derivative is a root of unity is said to be *rationally neutral*. A fixed point z_0 is said to *attract* some point $w \in \mathbb{C}_\infty$ if the iterates of r at w converge to z_0. We will be using the following result, whose proof can be found in [CG 93], Chapter III, Theorems 2.2 and 2.3:

Fact 1. *Let r be a rational function with $\deg r > 1$. If z_0 is an attracting or rationally neutral fixed point, then z_0 attracts some critical point of r.*

We will also use a version of the argument principle for harmonic functions. Let f be harmonic in an open set. We say that z is in the *critical set* of f if the Jacobian of f vanishes at z. In particular, in a sufficiently small disk containing z, there exist analytic functions h and g such that $f = h + \overline{g}$ in this disk. Then z is in the critical set of f if $|h'(z)|^2 - |g'(z)|^2 = 0$. We say that z is a *singular zero* of f if z is in the critical set of f and $f(z) = 0$. f is said to be *sense-preserving* at z if $|h'(z)| > |g'(z)|$ and is said to be *sense-reversing* at z if $|h'(z)| < |g'(z)|$. The *order* of a non-singular zero is positive if f is sense-preserving at the zero and negative if f is sense-reversing.

Let f be harmonic in a punctured neighborhood of z_0. We will refer to z_0 as a *pole* of f if $f(z) \to \infty$ as $z \to z_0$. Let C be an oriented closed curve that contains neither zeros nor poles of f. The notation $\Delta_C \arg f(z)$ denotes the increment in the argument of $f(z)$ along C. Following [ST 00], the *order* of a pole of f is given by $-\frac{1}{2\pi}\Delta_C \arg f(z)$, where C is a sufficiently small circle around the pole. We note that if f is sense-reversing on a sufficiently small circle around the pole, then the order of the pole will be negative. We will use the following version of the argument principle which is taken from [ST 00]:

Fact 2. *Let f be harmonic, except for a finite number of poles, in a simply connected domain D in the complex plane. Let C be a Jordan curve contained in D not passing through a pole or a zero, and let Ω be the open bounded region created by C. Suppose that f has no singular zeros in D and let N be the sum of the orders of the zeros of f in Ω. Let M be the sum of the orders of the poles of f in Ω. Then $\Delta_C \arg f(z) = 2\pi N - 2\pi M$.*

We note that a more general form of the argument principle can be found in [SS 02].

4. Non-repelling fixed points

Proposition 1. *Let p and q be relatively prime analytic polynomials. If $r = p/q$ is a rational function of degree $n > 1$, then the set of points for which $z = \overline{r(z)}$ and $|r'(z)| \leq 1$ has cardinality at most $2n - 2$.*

Proof. Let n_+ denote the number of points z_0 satisfying the conditions of Proposition 1. Following the approach of D. Khavinson and G. Świątek [KS 03], we consider the function $Q(z) = \overline{r(\overline{r(z)})}$, which is a rational function of degree n^2. As in [KS 03], we note that all fixed points of $\overline{r(z)}$ that are critical points for $f(z) = \overline{r(z)} - z$ are rationally neutral fixed points for $Q(z)$, so Fact 1 will apply. Their outline (Lemmas 1 - 3 in [KS 03] together with Fact 1) carry over word for word to the rational case if $\mathbb{C}$ is replaced by the Riemann sphere $\mathbb{C}_\infty$. In particular, each point z_0 which satisfies the conditions of Proposition 1 attracts at least $n+1$ critical points of Q.

Since deg $Q = n^2$, by the Riemann-Hurwitz relation (see [Fo 81], Section 17.14), Q has $2n^2 - 2$ critical points (counted with multiplicities) in $\mathbb{C}_\infty$. n_+ will be largest when all of the critical points of Q are attracted to points z_0 satisfying the conditions of Proposition 1. Since different fixed points attract disjoint sets of critical points, we see that $2n^2 - 2 \geq (n+1)\, n_+$ and the claim follows. □

5. Proof of Theorem 1

As in [KS 03], we say that the rational function $r(z)$ is *regular* if no critical point of $f(z) = \overline{r(z)} - z$ is a zero of f. This will allow us to apply Fact 2 to count the zeros of f.

Ansatz 1. *If r is regular of degree $n > 1$, then $f(z) = \overline{r(z)} - z$ has at most $5n - 5$ distinct finite zeros .*

Proof. Let $r = p/q$, where p and q are relatively prime, analytic polynomials. Let $n_p =$ deg p and $n_q =$ deg q. Then $n = \max\{n_p, n_q\}$. Let n_- denote the number of sense-reversing zeros of f and let n_+ be defined as in the proof of Proposition 1. Since r is regular, no zero of f lies on the critical set of f; this will allow us to apply Fact 2. In particular, n_+ counts the number of sense-preserving zeros of f. Then f has $n_+ + n_-$ zeros in the finite complex plane, counting multiplicity. Note that f is sense-reversing at each of the n_q poles of r; hence, the order of each pole is negative.

Consider the increment in the argument of f in a region bounded by a circle of large enough radius so that all of the finite zeros and poles of f are enclosed. We will apply Fact 2 and use our bounds on n_+ from Proposition 1. Note that we are counting zeros with multiplicity. We consider two cases.

(1) Assume that $n = n_q \geq n_p \geq 0$. Then the critical set of f is bounded. For z large, $r(z)$ is at most $O(1)$. Thus f is sense-preserving on our circle with an argument change of 2π. We also note that ∞ is not a zero of f. By Fact 2, $1 = (n_+ - n_-) - (-n) \leq 2n - 2 - n_- + n$. Hence $n_- \leq 3n - 3$, so f has at most $5n - 5$ zeros in $\mathbb{C}_\infty$ and the claim follows.

(2) Assume that $n = n_p \geq n_q + 1$. Since p and q are relatively prime, at least one of the two polynomials has a non-zero constant term.

We first suppose that p has a non-zero constant term. Then $f(0) \neq 0$. Let $z = 1/w$ and consider $F(w) = 1/\overline{r(1/w)} - w = \overline{G(w)/H(w)} - w$. Then F satisfies the conditions of the previous case, replacing n_q by deg H and n_p by deg G. Thus, F has at most $5n - 5$ zeros. Since $f(0) \neq 0$, f must also have at most $5n - 5$ zeros in the finite complex plane.

We now suppose that p does not have a non-zero constant term. Since p and q are relatively prime, the lowest order term in q must be a non-zero constant. Consider

$f_c(z) = \overline{(p(z)+c)/q(z)} - z = \overline{r_c(1/w)} - 1/w$. For all c sufficiently small, we have that $p(z)+c$ and q are relatively prime. In that case, f and f_c have the same poles and f_c approaches f uniformly as $c \to 0$ on compact subsets of $\mathbb{C}$ which do not contain any of the poles of f. We then substitute $z = 1/w$ to form $F_c(w) = 1/\overline{r_c(1/w)} - w = \overline{G_c(w)/H_c(w)} - w$. By construction, deg $G_c =$ deg $H_c = n$, so F_c has at most $5n-5$ zeros in the finite plane. Hence, this also holds for f_c since $f_c(0) \neq 0$.

Now suppose that f has more than $5n-5$ zeros in $\mathbb{C}$. By constructing a sufficiently small circle around each zero of f, we can guarantee that f is harmonic in each of the resulting closed disks. Moreover, we can make our circles sufficiently small so that each closed disk contains no other zeros of f and contains no critical points of f (recall that r is regular). Assume that f_c does not vanish in such a disk surrounding a zero, so $|f_c| > a > 0$. Then, for c sufficiently small, $\arg f = \arg(f_c(1+(f-f_c)/f_c))$ has zero increment around the circle bounding this disk, by Fact 2; this contradicts f having a zero in this disk. We see that f_c must have the same number of zeros as f in that disk (counting multiplicity). Thus, f_c must have more than $5n-5$ zeros in the finite plane, a contradiction. □

It remains to show that it is enough to consider regular rational functions.

Lemma 1. *If $r(z)$ is a rational function of degree greater than 1, then the set of complex numbers c for which $r(z) - c$ is regular is open and dense in $\mathbb{C}$.*

Proof. As in [KS 03], it is enough to show that the image of the critical set of $f(z) = \overline{r(z)} - z$ is nowhere dense in $\mathbb{C}$. In contrast to the polynomial case, it is possible for the critical set to be unbounded. This difficulty however is easily resolved by restricting f to an increasing family of concentric balls whose radii run to ∞. □

Lemma 1 shows that the set of regular rational functions is dense in the topology of uniform convergence in the spherical metric. As at the end of the proof of the Ansatz, it is easily seen that every zero of the function $f(z) = \overline{r(z)} - z$ must be a limit point for zeros of $\overline{r(z)} - z - c$ where $c \to 0$. Thus, Lemma 1 and the Ansatz prove the theorem.

6. Final remarks

Our proof does not indicate whether the $5n-5$ bound is always sharp. We have, however, found an example where this bound is attained for the case $n = 2$; namely, $f(z) = (\overline{z}^2 + \overline{z} - \frac{1}{2})/(\overline{z}^2 - \frac{3}{2}\overline{z} + 1) - z$. This function has $5n-5 = 5$ distinct finite zeros. This function has two poles: $z = \frac{1}{4}(3 - i\sqrt{7})$ and $z = \frac{1}{4}(3 + i\sqrt{7})$; f is sense-reversing at these poles. There are three sense-reversing zeros: $z = \frac{1}{2}$, $\frac{1}{2}(1 + i\sqrt{11})$, and $\frac{1}{2}(1 - i\sqrt{11})$. As expected from Fact 2 for an overall index change of $+1$ on large circles, there are two sense-preserving zeros; namely, $z = 1 - \sqrt{2}$ and $1 + \sqrt{2}$, Figure 1 is a Mathematica plot of the critical set of this function and Figure 2 shows its image. We note that f is sense-preserving in the unbounded component in Figure 1, sense-reversing in the larger of the two bounded components, and sense-preserving in the smaller of the two bounded components. We also note that f cannot be rewritten to model a 2-point gravitational lens.

L. Geyer [Ge 03] has recently shown that the $3n-2$ bound on the number of zeros of $f(z) = p(z) - \overline{z}$ where deg $p = n$ is sharp for all $n > 1$. D. Bshouty and A. Lyzzaik [BL 03]

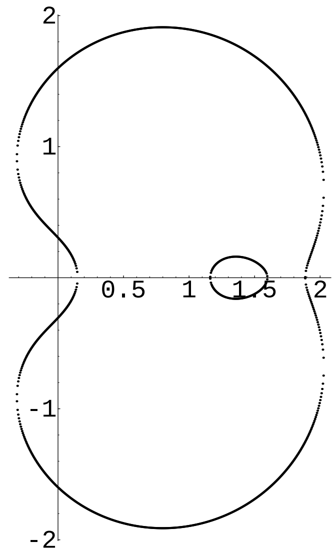

FIGURE 1. Critical set for $f(z) = \frac{\overline{z}^2+\overline{z}-\frac{1}{2}}{\overline{z}^2-\frac{3}{2}\overline{z}+1} - z$

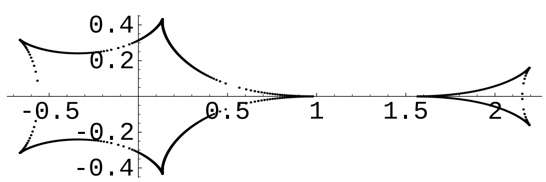

FIGURE 2. Image of critical set for $f(z) = \frac{\overline{z}^2+\overline{z}-\frac{1}{2}}{\overline{z}^2-\frac{3}{2}\overline{z}+1} - z$

have recently given an elementary proof for $n = 4, 5, 6, 8$. Hence, a sharp bound on the number of zeros of $f(z) = \overline{r(z)} - z$ must be at least $3n - 2$.

Results from gravitational lensing help with the question of sharpness of the $5n - 5$ bound in Theorem 1. We restrict our attention to the $\gamma = 0$, $\sigma > 0$ case of the lens equation (*cf.* Introduction). Mao, Petters, and Witt [MPW 97] have shown there are $3n + 1$ images when the light source is at the origin and the $n > 2$ point masses (each of mass $1/n$) are vertices of a regular polygon centered at the origin. Thus, in contrast to the case where $r(z)$ is a polynomial, by moving the poles of $r(z)$ into the finite plane, we see that a sharp bound on the number of zeros must be at least $3n + 1$. Further, by a clever perturbation argument, Rhie [Rh 03] has shown that the $5n - 5$ bound is attained and is hence sharp for all $n > 1$.

It is known that the number of lensed images is odd in the case of gravitational lensing by a regular gravitational lens ([St 97], [Bu 81]), given that the position of the source is not a critical value of the lens equation. (A regular lens has a smooth mass distribution. A point w is a critical value of the lens equation if the Jacobian of the lens mapping defined by the lens equation vanishes at any of the points in $f^{-1}(w)$.) Given that $5n - 5$ is not always odd, it may seem surprising that $5n - 5$ lensed images may be possible for an n-point gravitational lens with

$\gamma = 0$ and $\sigma > 0$. However, it was shown by Petters [Pe 92] using Morse theory, that this is not a problem. We summarize this result in the following corollary and present an alternate proof.

Corollary 2. *Suppose that we have an n-point gravitational lens modeled as above with $\gamma = 0$ and a light source which is not located at a critical value of the lens equation. Then the number of lensed images must be even when n is odd and odd when n is even.*

Proof. For this case of the lens equation, we may apply step (1) in the proof of the Ansatz. In particular, the number of images N will be $n_+ + n_-$, where n_+ denotes the number of sense-preserving solutions and n_- the number of sense-reversing solutions of the lens equation. From step (1), we see that $n_+ = 1 + n_- - n$, so $N = 1 + 2n_- - n$. $\square$

This result has also been shown by other techniques in [Rh 01] (footnote 2, page 2), which extends the approach of W. Burke [Bu 81] for the case of a regular gravitational lens to that of an n-point gravitational lens. Note that the above proof is similar to the approach used by N. Straumann [St 97] for the case of a regular gravitational lens.

We also note that Theorem 1 can be applied to the case of a more general mass distribution than point masses. For a compactly supported mass distribution with the projected mass density $d\psi$ in the lens plane, the lens equation transforms into $w = z + \gamma\overline{z} - sign(\sigma) \int_{\mathbb{C}} d\psi(\zeta) \,/\, (\overline{z} - \overline{\zeta})$ (*cf.* [NB 96] and [St 97]). In particular, we have the following corollary:

Corollary 3. *Suppose that the projected mass density of a gravitational lens consists of $n > 1$ radially symmetric, continuous, compactly supported densities in the lens plane. If the shear $\gamma = 0$, then the number of lensed images outside the support of the masses cannot exceed $5n - 5$. If the shear $\gamma \neq 0$, then the number of lensed images outside the support of the masses cannot exceed $5n$.*

Proof. Consider the integral term in the lens equation for one of the mass densities. A calculation shows that this integral evaluated for any z outside of the support of this radially symmetric mass density equals $b/(z - a)$, where a is the center of the mass density and b is a finite constant. Hence, for z outside of the support of the masses, the lens equation reduces to the lens equation for n point masses (each mass is reduced to a point mass at the sphere's center). Corollary 1 can then be applied to count the possible images, giving the upper bounds on images outside the support of the masses as claimed. $\square$

Remark 1. *If the mass distribution is composed entirely of luminous matter, then the only lensed images that will be visible are those that lie outside of the support of the masses. However, there are mass distributions that are not luminous (see* [APOD 01] *and* [APOD 03]*). For such mass distributions, it is possible to have lensed images that lie inside the support of the masses.*

A question related to bounding the number of zeros of $f(z) = \overline{r(z)} - z$ would be to find a bound on the number of zeros of $f(z) = R(z) - \overline{r(z)}$, where both R and r are rational functions. It is not clear what condition would be needed for such a function to have a finite number of zeros, much less a bound on the number of distinct zeros (beyond the obvious Bézout Theorem's bound when we assume that there are a finite number of zeros). For example, let $p(z)$ be a polynomial (of degree at least two) and consider $f(z) = p(z) - 1/\overline{p(z)}$. The zeros of this function form a lemniscate in the complex plane. In other words, $f(z)$ has an infinite number of zeros and these zeros are not isolated. Moreover, $\lim_{z\to\infty} f(z) = \infty$. If we compare this with

the result of Wilmshurst [Wil 98] for the case when f is entire: $f(z) \to \infty$ as $z \to \infty$ implies a finite number of zeros, we see that all of the zeros for an entire function will be isolated under Wilmshurst's condition for finite valence. Thus, f having finite poles makes the problem of discreteness of its zero set much more subtle.

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